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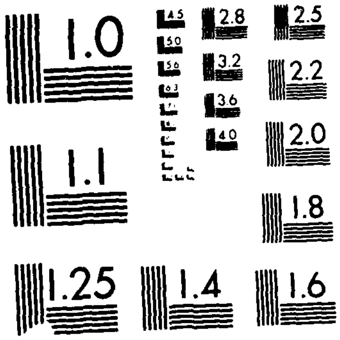

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AN APPROACH FOR CONSTRUCTING FAMILIES OF HOMOGENIZED EQUATIONS FOR PERIODIC MEDIA:

II: Properties of the kernel

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Abstract.

The paper is the second in the series devoted to the study of constructions of families of homogenizations. In the first paper [8] the properties of the kernel $\Phi(\cdot,h,t)$ were utilized. In this paper these properties are established.

1. Introduction.

In [8] we developed an integral representation of the solution to a differential equation that models the equations that arise in the study of periodic media (e.g., composite materials). The elliptic differential equation studied in [8] is

$$-\sum_{p,q=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\left(\frac{x}{h}\right)\frac{\partial u^h}{\partial x_q}(x)\right) + a_0\left(\frac{x}{h}\right)u^h(x) = f(x)$$

on $\mathbb{R}^n$, in which a_{pq} and a_0 are real-valued 2π-periodic functions and h is a given positive number. An alternate proof of the classical homogenization result (the limit of u^h as $h \to 0$) was given in [8], based on the integral formula for u^h that was developed there

The integral representation of u^h depends on the 2π-periodic function $\phi(\cdot,h,t)$ that satisfies

$$-e^{-iht\cdot y}\sum_{p,q=1}^{n} \frac{\partial}{\partial y_p}\left(a_{pq}(y)\frac{\partial}{\partial y_q}(\phi(y,h,t)e^{iht\cdot y})\right) + h^2 a_0(y)\phi(y,h,t) = h^2$$

on $\{y \in \mathbb{R}^n : |y_p| < \pi\}$, in which $t \in \mathbb{R}^n$. The main emphasis of [7] was placed on the function u^h; the properties of ϕ that were needed there were stated without proof. This paper presents an analysis of ϕ in order to prove these claims, namely, Lemmas 1 and 12 in [8]. Theorem 1 in this paper is equivalent to Lemma 1 in [8], whereas the content of Lemma 12 and the discussion preceding it in [8] is contained in Theorems 16 and 18 here.

In section 2, the notation used here and the equation that $\phi(\cdot,h,t)$ satisfies are given along with the statement of Theorem 1. The proof of Theorem 1 is presented in section 3. The expansion of $\phi(\cdot,h,t)$ in powers of h and properties of this expansion are developed in section 4. Section 5 is

devoted to developing several analyticity results associated with families of sesquilinear forms. The results of section 5 are used extensively in section 3. Additional details and references can be found in [7].

A method for systematically developing classes of differential equations, or even pseudodifferential operators, that describe the behavior of composite materials with a periodic structure has been introduced in [3] and is based on the results of [8] and this paper.

2. Notation and statement of the problem.

Let $S \equiv \{y \equiv (y_1,\ldots,y_n) \in \mathbb{R}^n : |y_k| < \pi \text{ for } k = 1,\ldots,n\}$, and for $j = 0,1$, denote the standard Sobolev norm on S by $\|\cdot\|_j$. In addition, define $|\cdot|_1$ by $|v_1| \equiv \left(\int_S \sum_{p=1}^{n} |\frac{\partial v}{\partial y_p}(y)|^2 dy\right)^{1/2}$. The Sobolev spaces of periodic functions for which S is the fundamental period, is denoted by $H^j_{per}(S)$, and is defined to be the completion with respect to $\|\cdot\|_j$, of th complex-valued, C^∞-functions on $\mathbb{R}^n$ that are 2π-periodic in each coordinate variable.

Let a_{pq}, for $p,q = 1,\ldots,n$, and a_0 be real-valued, 2π-periodic, L_∞-functions defined on $\mathbb{R}^n$. Furthermore, assume $a_{qp} = a_{pq}$ and that there exist positive constants γ_0 and γ_1 such that

$$
(1) \qquad \begin{cases} a_0(x) \geq \gamma_0 \quad \text{and} \\ \sum_{p,q=1}^{n} a_{qp}(x)\zeta_q\bar{\zeta}_p \geq \gamma_1 \sum_{p=1}^{n} |\zeta_p|^2 \quad \text{for all } \zeta_p \in \mathbb{C}, \end{cases}
$$

almost everywhere on $\mathbb{R}^n$. For each $h \in \mathbb{C}$ and $t \in \mathbb{C}^n$, define the sesquilinear form $\Phi(h,t) : H^1_{per}(S)\times H^1_{per}(S) \to \mathbb{C}$ by

$$\Phi(h,t)[\phi,v] = \int_S \left\{ \sum_{p,q=1}^{n} a_{pq}(y)\frac{\partial}{\partial y_q}(\phi(y)e^{iht\cdot y})\frac{\partial}{\partial y_q}(\overline{v(y)}e^{-iht\cdot y}) + h^2 a_0(y)\phi(y)v(y)\right\}dy. \tag{2}$$

In section 3 we will prove

<u>Theorem 1</u>. There exists a neighborhood $\hat{G}$ of $\mathbb{R}^{n+1}$ (contained in $\mathbb{C}^{n+1}$), such that a unique function $\phi(\cdot,h,t) \in H^1_{per}(S)$ exists for each $(h,t) \in \hat{G}$ and satisfies

$$\Phi(h,t)[\phi(\cdot,h,t),v] = h^2\int_S \overline{v(y)}dy \quad \text{for all} \quad v \in H^1_{per}(S). \tag{3}$$

Furthermore, the mapping

$$(h,t) \longmapsto \phi(\cdot,h,t) \in H^1_{per}(S) \tag{4}$$

is holomorphic on $\hat{G}$ (see Definition 19, section 5).

In the proof of Theorem 1, the following eigenvalue problem will be considered: seek $\lambda(h,t) \in \mathbb{C}$ and a non-zero function $\psi(\cdot,h,t) \in H^1_{per}(S)$ such that

$$\Phi(h,t)[\psi(\cdot,h,t),v] = \lambda(h,t)\int_S \psi(y,h,t)\overline{v(y)}dy \quad \text{for all} \quad v \in H^1_{per}(S). \tag{5}$$

Before proceeding to the proof of Theorem 1, we give two lemmas that we will use repeatedly.

<u>Lemma 2</u>. A constant C_0 exists such that

$$\frac{1}{C_0(1+\|t\|)}\|v\|_1 \le \|ve^{it\cdot y}\|_1 \le C_0(1+\|t\|)\|v\|_1$$

for all $v \in H^1(S)$, the standard Sobolev space, and for all $t \in \mathbb{R}^n$, where

$\|t\|^2 \equiv t_1^2 + \dots + t_n^2$.

Proof. The proof of the right-hand inequality is straight-forward. The inequality on the left is proved by applying the right-hand inequality to the function $w = ve^{it\cdot y}$:

$$\|v\|_1 = \|we^{-it\cdot y}\|_1 \le C_0(1+\|t\|)\|w\|_1. \qquad \square$$

Lemma 3. Let H be a complex Hilbert space with norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$, and let $\Phi : H\times H \longrightarrow \mathbb{C}$ be a sesquilinear form (i.e., $\Phi[\phi,v]$ is linear in ϕ and conjugate-linear in v). If there exist constants M and γ such that

$$|\Phi[\phi,v]| \le M\|\phi\|_H\|v\|_H$$

and

$$\gamma\|v\|_H^2 \le |\phi[v,v]|$$

for all ϕ and v in H, then for each $f \in H^*$, the space of bounded conjugate-linear functionals on H, there is a unique $\phi \in H$ such that

$$\Phi[\phi,v] = f(v) \quad \text{for all} \quad v \in H.$$

Moreover, $\|\phi\|_H \le \frac{1}{\gamma}\|f\|_{H^*}$.

Lemma 3 is known as the Lax-Milgram theorem (see [2]). The essense of the proof of Lemma 3 is the existence of a bounded operator A that maps H isomorphically onto H such that $\Phi[\phi,v] = (A\phi,v)_H$ for all ϕ and v in H. This fact will be used in the proof of Theorem 22, in section 5.

3. Proof of Theorem 1.

The ideas and results of section 5 will be used extensively in this section. Note that $H^0_{per}(S)$ and $H^1_{per}(S)$ satisfy the conditions imposed on H and V, respectively, in section 5, i.e., $H^1_{per}(S)$ is a continuously, densely, and compactly embedded subspace of $H^0_{per}(S)$. (A discussion of spaces of periodic functions is contained in [1].) Clearly, $(h,t) \in \mathbb{C}^{n+1} \longmapsto \Phi(h,t)[\phi,v] \in \mathbb{C}$ is an analytic function for each ϕ and v in $H^1_{per}(S)$.

We start here by determining an open set $G \subset \mathbb{C}^{n+1}$ such that $\mathbb{R}^{n+1} \subset G$ and such that $\Phi(h,t)$ satisfies inequalities similar to (34) and (35) for each $(h,t) \in G$. Then we will show that, in the sense of (5), 0 is not an eigenvlaue of $\Phi(h,t)$ when $h \neq 0$ and $(h,t) \in \mathbb{R}^{n+1}$, but that 0 is a simple eigenvalue of $\Phi(0,t)$. The conclusions of Theorems 26 and 27 in section 5 get us part of the way through the proof of theorem 1; we must investigate further the eigenvalue problem associated to $\Phi(0,t)$.

Lemma 4. There exists an open set $G \subset \mathbb{C}^{n+1}$ and real-valued functions M, γ, and μ such that μ is continuous on G and for each $(h,t) \in G$, $M(h,t) > 0$, $\gamma(h,t) > 0$,

i) $|\Phi(h,t)[\phi,v]| \le M(h,t)\|\phi\|_1\|v\|_1$ for all ϕ and v in $H^1_{per}(S)$,

and

ii) $\gamma(h,t)\|v\|_1^2 \le \mathrm{Re}(\Phi(h,t)[v,v]) + \mu(h,t)\|v\|_0^2$ for all $v \in H^1_{per}(S)$.

Moreover, G can be chosen so that $\mathbb{R}^{n+1} \subset G$ and so that $(\bar{h},\bar{t}) \in G$ whenever $(h,t) \in G$.

Proof. For each $z \in \mathbb{C}^n$ and ϕ and v in $H^1_{per}(S)$, define the sesquilinear form

$$B(z)[\phi,v] = \int_S \sum_{p,q=1}^{n} a_{pq}(y)\frac{\partial}{\partial y_q}(\phi(y)e^{iz\cdot y})\frac{\partial}{\partial y_q}(\overline{v(y)}e^{-iz\cdot y})dy;$$

thus, $\Phi(h,t)[\varphi,v] = B(ht)[\phi,v] + h^2\int_S a_0(y)\phi(y)\overline{v(y)}dy$. Whenever it is convenient in this proof, we will use $ht = \rho + i\sigma$ where ρ and σ are real n-vectors. Defining

$$F(\rho,\sigma)[\phi,v] = B(\rho+i\sigma)[\phi,v] - B(\rho)[\phi,v]$$

$$= \int_S \sum_{p,q=1}^{n} a_{pq}(y)\{\sigma_p \frac{\partial}{\partial y_q}(\phi(y)e^{i\rho\cdot y})\overline{v(y)}e^{-i\rho\cdot y}$$

$$- \sigma_q\phi(y)e^{i\rho\cdot y}\frac{\partial}{\partial y_q}(\overline{v(y)}e^{-i\rho\cdot y}) - \sigma_q\sigma_p\phi(y)\overline{v(y)}\}dy$$

it follows, since each a_{pq} is an L_∞-function, that there exists a constant K such that

$$(6) \qquad |F(\rho,\sigma)[\phi,v]| \le K\|\sigma\|(1+\|\sigma\|)\|\phi e^{i\rho\cdot y}\|_1\|ve^{i\rho\cdot y}\|_1.$$

In addition, there is a constant K' such that

$$|B(\rho)[\phi,v]| \le K'|\phi e^{i\rho\cdot y}|_1|ve^{i\rho\cdot y}|_1.$$

Using Lemma 2 and the fact that a_0 is bounded, it follows that there exists a positive number $M(h,t)$ for each (h,t) $\mathbb{C}^{n+1}$ such that (i) is true.

Next a consequence of (1) is that

$$B(\rho)[v,v] \ge \gamma_1|ve^{i\rho\cdot y}|_1^2$$

and therefore

$$(7) \qquad \mathrm{Re}(B(\rho+i\sigma)[v,v]) + \gamma_1\|v\|_0^2 = B(\rho)[v,v] + \mathrm{Re}(F(\rho,\sigma)[v,v]) + \gamma_1\|v\|_0^2$$

$$\ge (\gamma_1 - K\|\sigma\|(1+\|\sigma\|))\|ve^{i\rho\cdot y}\|_1^2.$$

We now define G by

$$G \equiv \{(h,t) \in \mathbb{C}^{n+1} : K\|\mathrm{Im}(ht)\|(1+\|\mathrm{Im}(ht)\|) < \frac{\gamma_1}{2}\}$$

and set

$$\gamma(h,t) = \frac{\gamma_1}{2C_0^2(1+\|\mathrm{Re}(ht)\|)^2}$$

in which C_0 is defined in Lemma 2. Finally, setting $K'' = \|a_0\|_{L_\infty(S)}$ and using (7) and Lemma 2, we conclude that

$$\begin{aligned}
&\mathrm{Re}(\Phi(h,t)[v,v]) + \gamma_1\|v\|_0^2 \\
&\quad = \mathrm{Re}(B(ht)[v,v]) + \mathrm{Re}(h^2\int_S a_o(y)|v(y)|^2dy) + \gamma_1\|v\|_0^2 \\
&\quad \le \gamma(h,t)\|v\|_1^2 - K''|h|^2\|v\|_0^2
\end{aligned}$$

for $(h,t) \in G$, which yields (ii) with $\mu(h,t) = K''|h|^2 + \gamma_1$. □

For each $(h,t) \in G$, we can associate to $\Phi(h,t)$ a closed operator $T(h,t)$ as in Theorem 22:

$$\Phi(h,t)[\phi,v] = (T(h,t)\phi,v)_{H^0_{per}(S)} \quad \text{for all} \quad v \in H^1_{per}(S)$$

and for all ϕ in the domain of $T(h,t)$, which is a dense subspace of $H^1_{per}(S)$. Reference is made to $T(h,t)$ in the next few paragraphs in order to draw upon the results of section 5.

For each $(h,t) \in \mathbb{R}^{n+1}$, a direct consequence of (1) and lemma 2 is

(8) $$\Phi(h,t)[v] \ge \gamma_1|ve^{iht\cdot y}|_1^2 + \gamma_0h^2\|v\|_0^2 \ge \frac{\min\{\gamma_0h^2,\gamma_1\}}{2C_0^2(1+\|(ht\|)^2}\|v\|_1^2$$

for all $v \in H^1_{per}(S)$. If $(h,t) \in \mathbb{R}^{n+1}$ and $h \neq 0$, then the hypotheses of Lemma 3 are satisfies; hence $\Phi(h,t)[\phi,v] = \int_S w(y)\overline{v(y)}dy$ for all $v \in H^1_{per}(S)$, is uniquely solvable for each $w \in H^0_{per}(S)$. Consequently, 0 is in the resolvent set of $T(h,t)$, and by Theorem 26, $\{(h,t) \in \mathbb{R}^{n+1} : h \neq 0\}$ is

contained in an open set in $\mathbb{C}^{n+1}$ on which (4) is holomorphic.

Now let $h = 0$. For any $\tau \in \mathbb{R}^n$ (or $\mathbb{C}^n$), we have from the first inequality in (8) that

$$\Phi(0,\tau)[v,v] \geq \gamma_1 |v|_1^2.$$

If 0 is an eigenvalue of $\Phi(0,\tau)$ with $\psi_0 \in H^1_{per}(S)$ being an associated eigenfunction, then

$$0 = \Phi(0,\tau)[\psi_0,\psi_0] \geq \gamma_1 |\psi_0|_1,$$

and consequently, ψ_0 is a constant function, which depends on τ. Therefore, 0 is an eigenvalue of (the associated closed operator) $T(0,\tau)$, and ψ_0 is the only eigenfunction associated to 0. There are no generalized eigenfunctions when $\tau \in \mathbb{R}^n$ because the identity $\Phi(0,\tau)[v,\phi] = \overline{\Phi(0,\tau)[\varphi,v]}$ implies, by Corollary 25, that $T(0,\tau)$ is selfadjoint.

The following conclusions can now be drawn from Theorem 27. For each $\tau \in \mathbb{R}^n$, there exists a neighborhood $G_\tau \subset G$ of $(0,\tau)$, and there exists a complex valued function λ, analytic on G_τ, sucht hat $\lambda(0,t) = 0$ for $(0,t) \in G_\lambda$ and such that $\lambda(h,t)$ is a simple eigenvalue of $\Phi(h,t)$ when $(h,t) \in G$. Furthermore, there exist two holomorphic functions $(h,t) \in G_\tau \longmapsto P(h,t)$ and $(h,t) \in G_\tau \longmapsto R_2(0,h,t)$, with values in the space of bounded linear operators that map $H^0_{per}(S)$ into $H^1_{per}(S)$, such that $P(h,t)$ projects $H^0_{per}(S)$ onto the 1-dimensional eigenspace spanned by the eigenvector associated to $\lambda(h,t)$, and such that

$$\phi(\cdot,h,t) = \frac{h^2}{\lambda(h,t)} P(h,t)\underset{\sim}{1} + h^2 R_2(0,h,t)\underset{\sim}{1} \tag{9}$$

for each $(h,t) \in G_\tau$ for which $\lambda(h,t) \neq 0$. Here $\underset{\sim}{1}$ is the constant function that takes on the value 1 on S. Note that $\lambda(h,t)$ is not identically 0 on G_τ since 0 is not an eigenvalue of $\Phi(h,\tau)$ when h is a non-zero

real number; therefore, this representation of $\phi(\cdot,h,t)$ is meaningful.

Clearly, the holomorphy of $(h,t)\longmapsto\phi(\cdot,h,t)$ on G_τ is completely determined by that of $\frac{h^2}{\lambda(h,t)}$. We will show that $\frac{h^2}{\lambda(h,t)}$ has an analytic continuation to $h = 0$, and that there is a suitable restriction G'_τ of G_τ, on which $\lambda(h,t) = 0$ if and only if $h = 0$. This will be done by determining part of the Taylor expansion of $\lambda(h,t)$ in powers of h about $h = 0$, where it will be seen that the coefficient of h^k is 0 for $k = 0,1$.

For the moment, we assume the existence of a function ω, analytic on G_τ, such that

$$\lambda(h,t) = h^2\omega(h,t) \quad \text{for} \quad (h,t) \in G_\tau. \tag{10}$$

Letting $\psi(\cdot,h,t) \in H^1_{per}(S)$ be the eigenfunction associated to $\lambda(h,t)$ as in (5), the first inequality in (8) yields

$$\gamma_0 h^2\|(\cdot,h,t)\|_0^2 \le \Phi(h,t)[\psi(\cdot,h,t),\psi(\cdot,h,t)] = \lambda(h,t)\|\psi(\cdot,h,t)\|_0^2$$

for $(h,t) \in G_\tau \cap \mathbb{R}^{n+1}$. Consequently, $\omega(h,t) \ge \gamma_0$ for $(h,t) \in G_\tau \cap \mathbb{R}^{n+1}$, and by continuity, there is an open set $G'_\tau \subset G_\tau$ such that $(G_\tau \cap \mathbb{R}^{n+1}) \subset G'_\tau$ and $\mathrm{Re}(\omega(h,t)) > 0$ for $(h,t) \in G'_\tau$. Thus $(h,t)\longmapsto\frac{h^2}{\lambda(h,t)} = \frac{1}{\omega(h,t)}$ is analytic on G'_τ, from which it follows that (9) is holomorphic on G'_τ as well. Assuming that (10) is valid, Theorem 1 is proven.

In the process of determining the Taylor expansion of $\lambda(h,t)$ in powers of h about $h = 0$, we develop a similar expansion of the eigenfunction $\psi(\cdot,h,t)$. In section 5 (see (45)), we show that $\psi(\cdot,h,t)$ can be chosen so that it depends holomorphically on $(h,t) \in G_\tau$. The Taylor expansion of $\lambda(h,t)$ and $\psi(\cdot,h,t)$ in powers of h will be obtained next, by expanding the eigenvalue equation (5) in powers of h and equating the coefficients of like powers.

Upon setting

$$
(11)\quad \begin{cases}
\Phi_0[\phi,v] \equiv \int_S \sum_{p,q=1}^{n} a_{pq}(y)\frac{\partial\phi}{\partial y_q}(y)\overline{\frac{\partial\phi}{\partial y_p}(y)}dy \\
\Phi_1(t)[\phi,v] \equiv i\int_S \sum_{p,q=1}^{n} a_{pq}(y)(\phi(y)\overline{\frac{\partial v}{\partial y_p}(y)}t_q - \frac{\partial\phi}{\partial y_q}(y)v(y)t_p)dy \\
\Phi_2(t)[\phi,v] \equiv \int_S\left[\sum_{p,q=1}^{n} a_{pq}(y)t_q t_p + a_0(y)\right]\phi(y)\overline{v(y)}dy
\end{cases}
$$

for all ϕ and v in $H^1_{per}(S)$ and for all $t \in \mathbb{C}^n$, it follows from the definition of $\Phi(h,t)$ that

$$
(12)\qquad \Phi(h,t) = \Phi_0 + \Phi_1(t)h + \Phi_2(t)h^2.
$$

Substituting the expansions $\lambda(h,t) = \sum_{k=0}^{n} \lambda_k(t)h^k$, $\psi(\cdot,h,t) = \sum_{k=0}^{n} \psi(\cdot,t)h^k$, and (12) into equation (5), equating like powers of h, and noting that $\lambda_0(t) = \lambda(0,t) = 0$, the following system of equations is derived:

$$
(13)\qquad \Phi_0[\psi_k(\cdot,t),v] = \begin{cases}
0, & k = 0 \\
\lambda_1(t)\int_S \psi_0(y,t)\overline{v(y)}dy - \Phi_1(t)[\psi_0(\cdot,t),v], & k = 1 \\
\sum_{\ell=1}^{k} \lambda_\ell(t)\int_S \Psi_{k-\ell}(y,t)\overline{v(y)}dy - \sum_{\ell=1}^{z} \Phi_\ell(t)[\psi_{k-\ell}(\cdot,t),v], & k \ge 2,
\end{cases}
$$

for all $v \in H^1_{per}(S)$. Successively solving the equations in (13) will yield formulas for the $\lambda_k(t)$ and $\psi_k(\cdot,t)$.

The method of solving (13) will be based on Lemma 6, below. First note that

$$
(14)\qquad \Phi_0[\phi,1] = \Phi_0[1,\phi] = \Phi_1(t)[1,1] = 0
$$

for all $\phi \in H^1_{per}(S)$ and for each $t \in \mathbb{C}^n$.

Definition 5. $W \equiv \{v \in H^1_{per}(S) : \int_S v(y)dy = 0\}$.

Each function in $H^1_{per}(S)$ can be represented uniquely as the sum of a constant function and a function in W.

Lemma 6. $|\cdot|_1$ is a norm on W, and there exists a constant K such that

$$|\Phi_0[\phi,v]| \le K|\phi|_1|v|_1$$
$$\Phi_0[v,v] \ge \gamma_1|v|_1^2$$

for all ϕ and v in W.

Each equation in (13) is of the form

$$\Phi_0[\psi_k,v] = F_k(v) \quad \text{for all} \quad v \in H^1_{per}(S),$$

in which F_k is a conjugate-linear form on $H^1_{per}(S)$ which depends on t; $\lambda_1(t),\ldots,\lambda_k(t)$; and $\psi_0(\cdot,t),\ldots,\psi_{k-1}(\cdot,t)$. Furthermore, F_k is bounded on $H^1_{per}(S)$. It is also bounded on W, as a result of the closed graph theorem (see (31)). A consequence of (14) is that we must ensure that $F_k(1) = 0$, which will determine $\lambda_k(t)$ uniquely. Then Lemmas 3 and 6 imply that $\psi_k(\cdot,t)$ is determined uniquely as an element in W, that is, up to an additive constant. However, the arbitrary constants in $\psi_k(\cdot,t)$ will remain essentially arbitrary because, being an eigenfunction, $\psi(\cdot,h,t)$ is uniquely determined up to a multiplicative constant only.

Theorem 7. For $k \ge 1$, define functions $\tilde{\chi}_k(\cdot,t) \in W$ according to

$$(15) \qquad \Phi_0[\tilde{\chi}_k(\cdot,t),v] = \begin{cases} -\Phi_1(t)[1,v], & k = 1 \\ -\Phi_1(t)[\tilde{\chi}_1(\cdot,t),v] - \Phi_2(t)[1,v], & k = 2 \\ \sum_{j=2}^{k-1}\lambda_j(t)\int_S \tilde{\chi}_{k-j}(y,t)\overline{v(y)}dy - \sum_{j=1}^{2}\Phi_j(t)[\tilde{\chi}_{k-j}(\cdot,t),v], & k \ge 3, \end{cases}$$

for all $v \in W$, where

$$(16)\qquad \lambda_\ell(t) = \begin{cases} 0, & \ell = 1 \\ \dfrac{1}{(2\pi)^n}(\Phi_1(t)[\tilde{\chi}_1(\cdot,t),1] + \Phi_2(t)[1,1]), & \ell = 2 \\ \dfrac{1}{(2\pi)^n}(\Phi_1(t)[\tilde{\chi}_{\ell-1}(\cdot,t),1] + \Phi_2(t)[\tilde{\chi}_{\ell-2}(\cdot,t),1]), & \ell \geq 3. \end{cases}$$

Next, define $\psi_\ell(\cdot,t) \in H^1_{per}(S)$ for $\ell \geq 0$ by

$$(17)\qquad \psi_\ell(\cdot,t) = \begin{cases} f_0(t), & \ell = 0 \\ \displaystyle\sum_{j=0}^{\ell-1} f_j(t)\tilde{\chi}_{\ell-j}(\cdot,t) + f_\ell(t), & \ell \geq 1, \end{cases}$$

in which each f_ℓ is a holomorphic function (in a neighborhood of $t = \tau$) and $f_0(t) \neq 0$ for all t. Then (16) and (17) solve (13).

<u>Proof</u>. Upon solving (13) with $k = 0$, we have that $\psi_0(\cdot,t)$ must be a constant function, which we denote by $f_0(t)$ and which we assume is non-zero for each t because $f_0(t) - \psi_0(\cdot,t) = \psi(\cdot,0,t)$ is an eigenfunction.

In order to simplify notation, we will drop the dependence on t in the remainder of the proof.

For $k = 1$, (13) becomes

$$\Phi_0[\psi_1,v] = \lambda_1 f_0 \int_S v(y)dy - f_0\Phi_1[1,v] \quad \text{for all} \quad v \in H^1_{per}(S).$$

Setting $v = 1$ and using (14) yields $0 = \lambda_1 f_0(2\pi)^n$, which implies $\lambda_1 = 0$ because $f_0 \neq 0$. Now (13) with $k = 1$ can be reduced to

$$\Phi_0[\psi_1,v] = -f_0\Phi_1[1,v] \quad \text{for all} \quad v \in W,$$

the solution of which is given by (17) with $\ell = 1$.

Substituting (16) and (17) into (13) with $k = 2$, yields

$$\Phi_0[\psi_2, v] = \lambda_2 f_0 \int_S \overline{v(y)}dy - (f_0\Phi_1[\tilde{\chi}_1, 1] + f_1\Phi_1[1, v]) - f_0\Phi_2[1, v]$$

for all $v \in H^1_{per}(S)$. Again, set $v = 1$ and use (14) to obtain $0 = \lambda_2 f_0(2\pi)^n - f_0(\Phi_1[\tilde{\chi}_1, 1] + \Phi_1[1, 1])$. Since $f_0 \neq 0$, the solution to this equation is given by (16). Now (13) with $k = 2$ can be reduced to

$$\Phi_0[\psi_2, v] = -f_0(\Phi_1[\tilde{\chi}_1, v] + \Phi_1[1, v]) - f_1\Phi_1[1, v] \quad \text{for all} \quad v \in W,$$

and the solution to this equation is given in (17).

An induction argument, similar to the one we will use in the proof of Theorem 15 establishes the remaining formulas in (16) and (17). □

As we noted earlier, the arbitrary nature of the constants $f_\ell(t)$ can be traced to the fact that the eigenfunction $\psi(\cdot, h, t)$ is determined uniquely, up to a multiplicative constant, only. To see this, let $c(h, t) = \sum_{k=0}^{\infty} c_k(t)h^k$ be analytic in h and t, with $c_0(t) \neq 0$, and define

$$f_k^*(t) = \sum_{\ell=0}^{k} p^{-\ell} c_\ell(t) f_{k-\ell}(t), \quad k \geq 0$$

and

$$\psi_k^*(\cdot, t) = \begin{cases} f_0^*(t), & k = 0 \\ \sum_{j=0}^{k-1} f_j^*(t)\tilde{\chi}_{k-j}(\cdot, t) + f_k^*(t), & k \geq 1. \end{cases}$$

Then $\sum_{k=0}^{\infty} \psi_k^*(\cdot, t)h^k$ is the power series expansion of $c(h, t)\psi(\cdot, h, t)$; clearly the form of ψ_k^* is the same as that of ψ_k.

We have shown that (10) is correct, and consequently, the proof of Theorem 1 is complete.

4. Expansion in powers of h.

The expansion of $\phi(\cdot,h,t)$ in powers of h can be determined formally by expanding (3) in powers of h and equating like powers. This formal process is valid because $h \longmapsto \phi(\cdot,h,t) \in H^1_{per}(S)$ is holomorphic at $h = 0$ for each $t \in \mathbb{C}^n$ such that $(0,t) \in \hat{G}$ (see Theorem 1), and because $\Phi(h,t)[\phi,v]$ is a polynomial in h and t for fixed ϕ and v in $H^1_{per}(S)$. By substituting

$$\phi(\cdot,h,t) = \sum_{k=0}^{\infty} \phi_k(\cdot,t)h^k \tag{18}$$

and (12) into equation (3), the following system of equations for the coefficients $\phi_k(\cdot,t)$ is obtained.

$$\Phi_0[\phi_k(\cdot,t),v] = \begin{cases} 0, & k = 0 \\ \Phi_1(t)[\phi_0(\cdot,t),v], & k = 1 \\ \int_S v(y)dy - \Phi_1(t)[\phi_{k-1}(\cdot,t),v] - \Phi_2(t)[\phi_{k-2}(\cdot,t),v], & k = 2 \\ -\Phi_1(t)[\phi_{k-1}(\cdot,t),v] - \Phi_2(t)[\phi_{k-2}(\cdot,t),v], & k \geq 3 \end{cases} \tag{19}$$

for all $v \in H^1_{per}(S)$. The radius of convergence of (18) depends on t, and each coefficient $\phi_k(\cdot,t)$ is in $H^1_{per}(S)$ and depends holomorphically on t.

The method of determining each ϕ_k is similar to that used in the proof of Theorem 7 to determine the Taylor expansions of $h \longmapsto \lambda(h,t)$ and $h \longmapsto \psi(\cdot,h,t)$. Recall that W is the subspace of $H^1_{per}(S)$ of functions that have an average value of 0. A consequence of (14) is that the right-hand side of (19) must be equal to 0 when $v = 1$. On the other hand, restricting v to be in W, Lemmas 3 and 6, applied to (19), uniquely determine $\phi_k(\cdot,t)$ as an element in W (i.e., up to an additive constant), in terms of

$\phi_{k-1}(\cdot,t)$ and $\phi_{k-2}(\cdot,t)$. Then $\phi_k(\cdot,t)$ becomes uniquely defined as an element in $H^1_{per}(S)$ by requiring the right-hand side of the equation for $\phi_{k+2}(\cdot,t)$ in (19) to be 0 when $v = 1$.

Theorem 15. For each $k \ge 1$ define $\chi_k(\cdot,t) \in W$ to be the solution of

$$(20) \qquad \Phi_0[\chi_k(\cdot,t),v] = \begin{cases} -\Phi_1(t)[1,v], & k = 1 \\ -\Phi_1(t)[\chi_1(\cdot,t),v] - \Phi_2(t)[1,v], & k = 2 \\ -\Phi_1(t)[\chi_{k-1}(\cdot,t),v] - \Phi_2(t)[\chi_{k-2}(\cdot,t),v], & k \ge 3 \end{cases}$$

for all $v \in W$, and for each $k \ge 0$ define $g_k(t) \in \mathbb{C}$ by

$$(21) \qquad g_k(t) = \begin{cases} \dfrac{(2\pi)^n}{\Phi_1(t)[\chi_1(\cdot,t),1]+\Phi_2(t)[1,1]}, & k = 0 \\ -\dfrac{g_0(t)}{(2\pi)^n} \displaystyle\sum_{j=0}^{k-1} g_j(t)(\Phi_1(t)[\chi_{k+1-j}(\cdot,t),1] + \Phi_2(t)[\chi_{k-j}(\cdot,t),1]), & k \ge 1. \end{cases}$$

Then the coefficient of h^k in (18) is given by

$$(22) \qquad \phi_k(\cdot,t) = \begin{cases} g_0(t), & k = 0 \\ \displaystyle\sum_{j=0}^{k-1} g_j(t)\chi_{k-j}(\cdot,t) + g_k(t), & k \ge 1. \end{cases}$$

Proof. Throughout this proof we will suppress all dependence upon t. First, each χ_k is well defined in W by (20) because of Lemmas 3 and 6 and because each right-hand side in (20) is a bounded conjugate-linear form on W (see (30) and (31)).

It follows immediately from (19) with $k = 0$, that ϕ_0 is a constant function, which we denote by g_0. For $k = 1$ in (19), we now have

$$\Phi_0[\phi_1,v] = -g_0\Phi_1[1,v] \quad \text{for all} \quad v \in H^1_{per}(S).$$

Since $\Phi_1[1,1] = 0$ (see (14)), the solution ϕ_1 has the form of (22); χ_1 is defined by (20), but g_0 and g_1 are arbitrary constants.

We next consider $k = 2$ in (19), and substituting (22), we obtain

$$(23) \qquad \Phi_0[\phi_2, v] = \int_S \overline{v(y)}dy - (g_0\Phi_1[\chi_1, v] + g_1\Phi_1[1, v]) - g_0\Phi_2[1, v]$$

$$\text{for all } v \in H^1_{per}(S).$$

Setting $v = 1$ and using (14) yields

$$0 = \Phi_0[\Phi_2, 1] = (2\pi)^n - g_0(\Phi_1[\chi_1, 1] + \Phi_2[1, 1]).$$

Solving for g_0 yields (21) with $k = 0$. On the other hand, requiring v to be in W, and using (20), gives

$$\begin{aligned}\Phi_0[\phi_2, v] &= -g_0(\Phi_1[\chi_1, v] + \Phi_2[1, v]) - g_1\Phi_1[1, v] \\ &= g_0\Phi_0[\chi_2, v] + g_1\Phi_0[\chi_1, v],\end{aligned}$$

which can be solved easily for ϕ_2 as a function in W. Thus, ϕ_0 is completely determined as in (22), whereas ϕ_1 and ϕ_2 have the form of (22); we have yet to show that g_1 and g_2 have been correctly defined.

Now let $k \geq 3$, and assume that $\phi_0, \ldots, \phi_{k-3}$ are given by (22), and that ϕ_{k-2} and ϕ_{k-1} have the form of (22). That is, we are certain that $g_0, \ldots, g_{k-3}$ are correctly defined in (21), but are not sure about g_{k-2} and g_{k-1}. We with to show that g_{k-2} is correctly defined in (21) and that ϕ_k has the form of (22). Making use of our assumptions, (19) becomes

$$\begin{aligned}\Phi_0[\phi_k, v] = &-\sum_{j=0}^{k-3} g_j(\Phi_1[\chi_{k-1-j}, v] + \Phi_2[\chi_{k-2-j}, v]) \\ &- g_{k-2}(\Phi_1[\chi_1, v] + \Phi_2[1, v]) - g_{k-1}\Phi_1[1, v].\end{aligned}$$

Setting $v = 1$ yields

$$0 = \Phi_0[\phi_k, 1] = -\sum_{j=0}^{k-3} g_j(\Phi_1[\chi_{(k-2)+1-j}, v] + \Phi_2[\chi_{(k-2)-j}, 1]) - \frac{(2\pi)^n}{g_0} g_{k-2},$$

which can be solved for g_{k-2}, and thus obtaining (21). Finally, upon requiring v to be in W, it follows from (20) that ϕ_k has the form given in (22). □

Next, we sufficiently investigate the properties of the expansion of $\phi(\cdot,h,t)$ in powers of h, to be able to prove Lemma 12 in [8]. The results are stated here in Theorems 16 and 18.

We begin by determining the dependence on t of $\chi_1(\cdot,t)$. Expanding the right-hand side of (20) with $k = 1$, accorrding to (11) yields

$$\Phi_0[\chi_1(\cdot,t), v] = -i\sum_{q=1}^{n}\left[\int_S \sum_{p=1}^{n} a_{pq}(y)\overline{\frac{\partial v}{\partial y_q}(y)}dy\right]t_q$$

for all $v \in W$. Now define $\chi_{1;q} \in W$, for each $q = 1,\dots,n$, to be the unique solution (cf. Lemmas 3 and 6) of

$$\Phi_0[\chi_{1;q}, v] = \int_S \sum_{p=1}^{n} a_{pq}(y)\overline{\frac{\partial v}{\partial y_q}(y)}dy \quad \text{for all} \quad v \in W. \tag{24}$$

Consequently,

$$_1(\cdot,t) = i\sum_{q=1}^{n}\chi_{1;q}t_q. \tag{25}$$

Note that $\chi_{1;q}$ is a real-valued function because the same is true for each a_{pq}. Furthermore, it follows from (24) and the definition of Φ_0 that

$$\Phi_0[\chi_{1;q} + y_q, v] = 0 \quad \text{for all} \quad v \in W. \tag{26}$$

The following formula for $g_0(t)$ can be easily obtained by substituting (11) and (25) into (21):

$$g_0(t) = \frac{1}{\sum_{p,q=1}^{n} A_{pq} t_q t_p + A_0}, \tag{27}$$

where

$$\begin{cases} A_0 \equiv \frac{1}{(2\pi)^n} \int_S a_0(y)dy \\ A_{pq} \equiv \frac{1}{(2\pi)^n} \int_S (a_{pq}(y) + \sum_{r=1}^{n} a_{pr}(y) \frac{\partial \chi_{1;q}}{\partial y_r}(y))dy. \end{cases} \tag{28}$$

Theorem 16. $A_0 \in \mathbb{R}$, $A_{pq} \in \mathbb{R}$, $A_{qp} = A_{pq}$, and $0 < g_0(t) \le \frac{1}{\gamma_1 \|t\|^2 + \gamma_0}$ for $t \in \mathbb{R}^n$.

Proof. A_0 and $\{A_{pq} : p,q = 1,\ldots,n\}$ are real numbers because each integrand in (28) is a real-valued function. It follows from the definition of Φ_0 and from the formula for A_{pq} that

$$A_{pq} = \frac{1}{(2\pi)^n} \Phi_0[\chi_{1;q} + y_q, y_p];$$

and upon using (26) with $v = \chi_{1;p}$ we obtain

$$A_{pq} = \frac{1}{(2\pi)^n} \Phi_0[\chi_{1;q} + y_q, \chi_{1;p} + y_p] \quad \text{for all} \quad p,q = 1,\ldots,n. \tag{29}$$

The symmetry of A_{pq} follows from (29) (and (11)), since symmetry conditions are imposed on the coefficients a_{pq}, and since each function involved in (29) is real-valued.

An immediate consequence of (1) and (28) is $A_0 \ge \gamma_0$. Next, define $\xi(y,t) = \sum_{q=1}^{n} t_q y_q$. Then (25), (26), Lemma 6, and the fact that $\chi_1(\cdot,t)$ is a periodic function imply

$$\sum_{p,q=1}^{n} A_{pq}t_q t_p = \frac{1}{(2\pi)^n}\Phi_0[-i\chi_1(\cdot,t)+\xi(\cdot,t),-i\chi_1(\cdot,t)+\xi(\cdot,t)]$$

$$\geq \frac{\gamma_1}{(2\pi)^n}|-i\chi_1(\cdot,t)+\xi(\cdot,t)|_1^2$$

$$= \frac{\gamma_1}{(2\pi)^n}\int_S \sum_{p=1}^{n} |-i\frac{\partial\chi_1}{\partial y_p}(y,t)+t_p|^2 dy$$

$$= \frac{\gamma_1}{(2\pi)^n}\{|\chi_1(\cdot,t)|_1^2 + 2\sum_{p=1}^{n} t_p\int_S \mathrm{Im}\left[\frac{\partial\chi_1}{\partial y_p}(y,t)\right]dy + (2\pi)^n\|t\|^2\}$$

$$\geq \gamma_1\|t\|^2.$$

Then, $\frac{1}{g_0(t)} \geq \gamma_1\|t\|^2 + \gamma_0$ follows from (27). □

Next we prove

Lemma 17. There are positive constants η and θ which are independent of $t \in \mathbb{R}^n$ such that

$$\|\phi_k(\cdot,t)\|_1 \leq \eta g_0(t)\theta^k(1+\|t\|)^k \quad \text{for each} \quad k \geq 0.$$

Proof. After stating a few preliminary results, the proof is presented in three steps. First, an upper bound for $|\chi_k(\cdot,t)|_1$ is derived from (20). This result is then used with (21) in order to obtain an upper bound on $|g_k(t)|$. Finally these two bounds and (22) will give an upper bound on $\|\phi_k(\cdot,t)\|_1$.

It follows from (11) that there is a constant c_1 such that

$$(30) \qquad |\Phi_k(t)[\phi,v]| \leq c_1(1+\|t\|)^k\|\phi\|_1\|v\|_1 \quad \text{for} \quad k = 1,2,$$

because the coefficients a_0 and a_{pq} are L_∞-functions. The closed graph theorem implies that there is a constant c_2, which we take to be larger than

$(2\pi)^{n/2}$, such that

(31) $$\|v\|_1 \le c_2|v|_1 \quad \text{for all} \quad v \in W.$$

We now have from (20) that

$$|\Phi_0[\chi_k(\cdot,t),v]| \le \begin{cases} c_1c_2^2(1+\|t\|)|v|_1 & k=1 \\ c_1c_2^2\{(1+\|t\|)|\chi_1(\cdot,t)|_1 + (1+\|t\|)^2\}|v|_1, & k=2 \\ c_1c_2^2\sum_{j=1}^{2}(1+\|t\|)^j|\xi_{k-j}(\cdot,t)|_1|v|_2, & k\ge 3. \end{cases}$$

Lemmas 3 and 6 imply

$$|\chi_k(\cdot,t)|_1 \le \begin{cases} c_e(1+\|t\|), & k=1 \\ c_3(|\chi_1(\cdot,t)|_1 + (1+\|t\|))(1+\|t\|), & k=2 \\ c_3(|\chi_{k-1}(\cdot,t)|_1 + (1+\|t\|)|\chi_{k-2}(\cdot,t)|_1)(1+\|t\|), & k\ge 3, \end{cases}$$

in which $c_3 \equiv \dfrac{c_1c_2^2}{\gamma_1}$. An induction argument proves

(32) $$|\chi_k(\cdot,t)|_1 \le (c_3+1)^k(1+\|t\|)^k \quad \text{for} \quad k=1,2,\dots\ .$$

Next, using (30), (31), and (32) in (21) yields

$$|g_k(t)| \le \frac{g_0(t)}{(2\pi)^n}\sum_{j=0}^{k-1}|g_j(t)|c_1c_2^2(1+\|t\|)\cdot(|\chi_{k+1-j}(\cdot,t)|_1 + (1+\|t\|)|\chi_{k-j}(\cdot,t)|_1)$$

$$\le \frac{c_1c_2^2(c_3+2)}{(2\pi)^n}g_0(t)(1+\|t\|)^2\sum_{j=0}^{k-1}|g_j(t)|(c_3+1)^{k-j}(1+\|t\|)^{k-j}$$

for $k \ge 1$. A consequence of Theorem 16 is that this last inequalily can be rewritten as

$$|g_k(t)| \le c_4 \sum_{j=0}^{k-1} |g_j(t)|(c_3+1)^{k-j}(1+\|t\|)^{k-j} \quad \text{for} \quad k \ge 1,$$

where

$$c_4 = \frac{2c_1c_2^2(c_3+2)}{(2\pi)^n \min\{\gamma_0, \gamma_1\}}.$$

Another induction argument then proves

$$(33) \qquad |g_k(t)| \le g_0(t)(c_3+1)^k(c_4+1)^k(1+\|t\|)^k \quad \text{for} \quad k \ge 0.$$

Finally, substitute (32) and (33) into (22) to obtain

$$\|\phi_k(\cdot,t)\|_1 \le \sum_{j=0}^{k-1} g_0(t)c_2(c_3+1)^k(c_4+1)^j(1+\|t\|)^k$$

$$+ g_0(t)c_2(c_3+1)^k(c_4+1)^j(1+\|t\|)^k$$

$$\le c_2g_0(t)(c_3+1)^k(c_4+2)^k(1+\|t\cap)^k,$$

which finishes the proof. □

By computing a majorizaing series for (18), the next theorem is a consequence of Lemma 17.

<u>Theorem 18</u>. Let η and θ be given as in Lemma 17, and suppose $h > 0$ and $t \in \mathbb{R}^n$ satisfy $\theta(1+\|t\|)h < 1$. Then

$$\|\phi(\cdot,h,t)\|_1 \le \frac{\eta g_0(t)}{1-\theta(1+\|t\|)h},$$

and

$$\|\phi(\cdot,h,t) - \sum_{j=0}^{k} \phi_j(\cdot,t)h^j\|_1 \le \frac{\eta\theta^{k+1}g_0(t)}{1-\theta(1+\|t\|)h}(1+\|t\|)^{k+1}h^{k+1}$$

for $k \ge 0$.

5. Appendix.

In this section we develop some of the theory that was used in section 3, when making some of our analyticity claims. The main goals here are Theorems 19 and 29. Throughout this section, we make the following assumptions. Let V and H be separable, complex Hilbert spaces in which V is a compactly and continuously embedded dense subspace of H. We denote the associated inner products by $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_H$, and the associated norms by $\|\cdot\|_V$ and $\|\cdot\|_H$. Let G be an open set in $\mathbb{C}^n$, and consider a family of sesquilinear forms $\Phi(z) : V\times V\longrightarrow\mathbb{C}$, defined for each $z \in G$. Suppose that there are real-valued functions M, γ, and μ defined on G such that $M(z) > 0$ and $\gamma(z) > 0$ for $z \in G$, μ is continuous on G, and for each $z \in G$

$$|\Phi(z)[\phi,v]| \le M(z)\|\phi\|_V\|v\|_V \quad \text{for all } \phi \text{ and } v \text{ in } V \tag{34}$$

and

$$\gamma(z)\|v\|_V^2 \le \mathrm{Re}(\Phi(z)[v,v]) + \mu(z)\|v\|_H^2 \quad \text{for all } v \in V. \tag{35}$$

Furthermore, suppose that $z\longmapsto\Phi(z)[\phi,v]$ is analytic on G, for each ϕ and v in V.

For a given $w \in H$, we want to determine the dependence on z, in particular situations, of $\phi(z) \in V$, which satisfies

$$\Phi(z)[\phi(z),v] = (w,v)_H \quad \text{for all } v \in V.$$

In so doing, we will consider the eigenvalue problem: seek $\lambda(z) \in \mathbb{C}$ and $\psi(z) \in V$ such that

$$\Phi(z)[\psi(z),v] = \lambda(z)(\psi(z),v)_H \quad \text{for all } v \in V.$$

Let $a \in G$. If 0 is not an eigenvalue of $\Phi(a)$, then we will show that $\Phi(z)$ exists for, and depends analytically on, z in a neighborhod of a. If

0 is a "simple" eigenvalue of $\Phi(a)$, then we will show that λ, with $\lambda(a) = 0$, is analytic in a neighborhood of a, and we will derive an expression for $\phi(z)$, exhibiting its dependence on $\lambda(z)$.

When z is one complex variable, many of the results of this section can be found in [5]. An important difference is that we have imposed alternate conditions ((34) and (35)) on $\Phi(z)$. This allows us to conclude that ϕ is analytic with values in V, rather than in H, which is the conclusoin in [5].

At this point we want to give a definition of analyticity, or holomorphy, for Banach space valued functions of several complex variables. Several definitions are possible. In a setting more general than Banach spaces, three definitions are stated and proven to be equivalent, in chapter III of [4]. First an open polydisc $\Delta(a,\rho)$ in $\mathbb{C}^n$ with center a and multiradius $\rho \equiv (\rho_1,\ldots,\rho_n)$, where $0 < \rho_j < \infty$, is defined by

$$\Delta(a,\rho) \equiv \{z \in \mathbb{C}^n : |z_j - a_j| < \rho_j \text{ for } j = 1,\ldots,n\}.$$

<u>Definition 19</u>. Let W be a Banach space, and recall that G is an open set in $\mathbb{C}^n$. A function $w : G \longrightarrow W$ is analytic, or holomorphic, if for each $a \in G$ there is a polydisc $\Delta(a,\rho) \subset G$ and a set of coefficients $\{w_\alpha(a) : \alpha$ is a multi-index$\} \subset W$ such that $\sum_{0\le|\alpha|} w_\alpha(a)(z-a)^\alpha$ converges in W to $w(z)$ for each $z \in \Delta(a,\rho)$.

We will have several occasions in which the next two lemmas will be used. When $n = 1$, proofs can be found in [5], and for the general case they are prove in [7]. Let H_1 and H_2 be two separable, complex Hilbert spaces and denote the inner product on H_2 by $(\cdot,\cdot)_2$. Denote the space of bounded linear operators mapping H_1 into H_2 by $B(H_1,H_2)$.

Lemma 20. Let $T(z) \in B(H_1, H_2)$ for each $z \in G$. The following statements are equivalent:

i) $T : G \longrightarrow B(H_1, H_2)$ is holomorphic;

ii) $T(\cdot)\phi : G \longrightarrow H_2$ is holomorphic for each $\phi \in H_1$;

iii) $(T(\cdot)\phi, v)_2 : G \longrightarrow \mathbb{C}$ is holomorphic for each $\phi \in H_1$ and $v \in H_2$.

Lemma 21. Suppose $T : G \longrightarrow B(H_1, H_2)$ is holomorphic, and let $a \in G$. If $T(a)^{-1} \in B(H_2, H_1)$, then there exists a neighborhood $G_a \subset G$ of a such that $T(z)^{-1}$ exists for $z \in G_1$ and $T(\cdot)^{-1} : G_1 \longrightarrow B(H_2, H_1)$ is holomorphic.

We are now ready to state and prove the results on which the main theorems of this section are based. We begin by showing that there exists a closed operator $T(z) : D(T(z)) \subset V \longrightarrow H$ such that $\Phi(z)[\phi, v] = (T(z)\phi, v)_H$ for all $\phi \in D(T(z))$ and $v \in V$. The next theorem gives one way of constructing such an operator, which will be convenient for us in what follows. Other forms of this representation theorem can be found in [5] and [6]. See also [2] and [9].

Throughout this section, we will denote the domain and range of an operator T by $D(T)$ and $R(T)$. Also, in the next theorem only, the dependence on z as we have stated so far, is inconsequential, and so we drop it.

Theorem 22. Let $\Phi : V \times V \longrightarrow \mathbb{C}$ be a sesquilinear form for which there exist real constants $M > 0$, $\gamma > 0$ and μ such that

(34′) $$|\Phi[\phi, v]| \le M\|\phi\|_V \|v\|_V \quad \text{for all } \phi \text{ and } v \text{ in } V$$

and

(35′) $$\phi\|v\|_V^2 \le R(\Phi[v, v]) + \mu\|v\|_H^2 \quad \text{for all } v \in V.$$

Then there is a unique closed operator $T : V \longrightarrow H$ such that

i) $D(T)$ is dense in V;

ii) $\Phi[\phi, v] = (T\phi, v)_H$ for all $\phi \in D(T)$ and $v \in V$; and

iii) given $\phi \in V$ and $w \in H$, if $\Phi[\phi, v] = (w, v)_H$ for all v in a dense subspace of V, then $\phi \in D(T)$ and $T\phi = w$.

Proof. Uniqueness follows from (iii); let there be another closed operator S such that $\Phi[\phi, v] = (S\phi, v)_H$ for all $\phi \in D(S)$ and $v \in V$. Then $\phi \in D(T)$ and $T\phi = S\phi$.

Since the embedding of V in H is continuous, it follows from the Riesz representation theorem that there is a linear operator $F \in B(H, V)$ such that

$$(w, v)_H = (Fw, v)_V \quad \text{for all} \quad w \in H \quad \text{and} \quad v \in V.$$

Furthermore, F is a 1-1 map, and $R(F)$ is dense in V.

Next, it follows from (34′) and (35′) that $\Phi[\cdot,\cdot] + \mu(\cdot,\cdot)_H$, as a sesquilinear form on $V \times V$, satisfies the hypotheses of Lemma 3. From the discussion that follows Lemma 3, it follows that there exists an operator $A_\mu \in B(V, V)$ such that $A_\mu^{-1} \in B(V, V)$ adn $\Phi[\phi, v] + \mu(\Phi, v)_H = (A_\mu \phi, v)_V$ for all ϕ and v in V. Now define

$$D(T) \equiv \{\phi \in V : A_\mu \phi \in R(F)\}$$

and set

$$T = (F^{-1} A_\mu - \mu I)_{D(T)},$$

in which I is the identity operator on H.

Clearly the choice of μ in (35′) is not unique. That the definitions of $D(T)$ and T are independent of μ can be seen, as follows. Let $\mu' \neq \mu$ be a real number for which (35′) remains valid when μ is replaced by μ'. (The value of $\gamma > 0$ makes no difference.) The definitions of A_μ and $A_{\mu'}$ imply

$$(A_{\mu'}\phi, v)_V - \mu'(F\phi, v)_V = (A_\mu\phi, v)_V - \mu(F\phi, v)_V$$

for all ϕ and v in V. Thus

$$(A_{\mu'} - \mu' F)\phi = (A_\mu - \mu F)\phi \quad \text{for all} \quad \phi \in V,$$

from which it follows that $A_{\mu'}\phi \in R(F)$ if and only if $A_\mu\phi \in R(F)$, and that

$$T = (F^{-1}A_\mu - \mu I)|_{D(T)} = (F^{-1}A_{\mu'} - \mu' I)|_{D(T)}.$$

Since A_μ is an isomorphism on V and $R(F)$ is dense in V, it follows that $D(T)$ is dense in V, which proves (i).

Statement (ii) follows from the definitions of A_μ and T.

To prove (iii), let $\phi \in V$ and $w \in H$ such that $\Phi[\phi, v] = (w, v)_H$ for all v in a dense subset of V. Then

$$(A_\mu\phi - \mu F\phi, v)_V = \Phi[\phi, v] = (W, v)_H = (Fw, v)_V$$

for all v in a dense subset of V, which implies $A_\mu\phi - \mu F\phi = Fw$ so that $\phi \in D(T)$ and $T\phi = w$.

Finally, $T : D(T) \subset V \longrightarrow H$ is a closed operator because $A_\mu^{-1}F \in B(H, V)$.□

Noting (34) and (35), Theorem 22 implies the existence of a unique closed operator $T(z) : V \longrightarrow H$ for each $z \in G$, such that $D(T(z))$ is dense in v; $\Phi(z)[\phi, v] = (T(z)\phi, v)_H$ for all $\phi \in D(T(z))$ and $v \in V$; and for any $w \in H$,

(36) $$T(z)\phi = w \text{ if and only if } \phi(z)[\phi, v] = (w, v)_H \text{ for all } v \text{ in a dense subset of } V.$$

Since $T(z) : D(T(z)) \subset V \longrightarrow H$ is closed, the resolvent operator $R(\zeta, z) \equiv (T(z) - \zeta)^{-1}$ belongs to $B(H, V)$ for each $\zeta \in \rho(T(z))$, the resolvent set of $T(z)$. A consequence of (34), (35), (36), and Lemma 3 is that $\rho(T(z))$ contains $\{\zeta \in \mathbf{C} : -\mathrm{Re}\ \zeta \geq \mu(z)\}$. A standard result in the spectral theory of

operators is that $\rho(T(z))$ is an open set in $\mathbb{C}$. In Theorem 24 below, we will prove that

(37) $$\mathcal{G} \equiv \{(\zeta,z) \in \mathbb{C}^{n+1} : \zeta \in \rho(T(z)) \text{ and } z \in G\}$$

is an open set also, and that $R : \mathcal{G} \to B(H,V)$ is holomorphic.

First we prove a preliminary result.

Lemma 23. For each $a \in G$, if $\zeta \in \rho(T(a))$ then there exists a neighborhood $G_{a\zeta} \subset G$, of a such that

i) $\zeta \in \rho(T(z))$ and $R(\zeta,z) \in B(H,V)$ for all $z \in G_{a\zeta}$, and

ii) $z \longmapsto R(\zeta,z) \in B(H,V)$ is holomorphic on $G_{a\zeta}$.

Proof. The continuity of μ allows us to choose a neighborhood $G_a \subset G$ of a, and a number $\mu_a \geq \mu(z)$ for $z \in G_a$; that is,

$$\gamma(z)\|v\|_V^2 \leq \mathrm{Re}(\Phi(z)[v,v]) + \mu_a\|v\|_H^2$$

for all $v \in V$ and $z \in G_a$. We will first prove the lemma for $\zeta = -\mu_a$ and then use the identity

(38) $$(T(z)-\zeta)r(-\mu_a,z) = I_H - (\zeta+\mu_a)R(-\mu_a,z) \quad \text{for} \quad z \in G_a,$$

to prove the lemma for arbitrary $\zeta \in \rho(T(a))$.

As in the proof of Theorem 22, associate an operator $A_a(z) \in B(V,V)$ to $\Phi(z)$ such that

(39) $$\Phi(z)[\phi,v] + \mu_a(\phi,v)_H = (A_a(z)\phi,v)_V$$

for all ϕ and v in V. It was shown there that while $A_a(z)$ depends on the choice of μ_a, $A_a(z) - \mu_a F$ does not depend on μ_a, where $F \in B(H,V)$ is defined by $(Fw,v)_V = (w,v)_H$ for all $w \in H$ adn $v \in V$. Furthermore, $D(T(z)) = \{\phi \in V : A_a(z)\phi \in R(F)\}$ and $T(z) = (F^{-1}A_a(z)-\mu_a)|_{D(T(z))}$ for $z \in G_a$. Consequently, $T(z) + \mu_a = F^{-1}A_a(z)|_{D(T(z))}$ is a one-to-one map of

$D(T(z))$ onto H, and it follows that $-\mu_a \in \rho(T(z))$ whenever $z \in G_1$. According to Lemma 20 and the hypothesis that $z \longmapsto \Phi(z)[\phi, v]$ is analytic, it follows from (39) that $A_a : G_a \longrightarrow B(V,V)$ is holomorphic. Since $A_a(z)^{-1} \in B(V,V)$ for each z in G_a, Lemma 21 implies that $A_a(\cdot)^{-1} : G_1 \longrightarrow B(V,V)$ is holomorphic. Therefore $z \longmapsto R(-\mu_a, z) = A_a(z)^{-1}F \in B(H,V)$ is holomorphic on G_a.

Now, $\zeta \in \rho(T(a))$. When $z = a$, the left-hand side of (38) is a one-to-one map of H onto H; hence its inverse exists and belongs to $B(H,H)$. As a function of z with values in $B(H,H)$, $R(-\mu_a, z)$ is holomorphic on G_a because it is holomorphic as a function with values in $B(H,V)$ and because the embedding of V into H is bounded. Thus the right-hand side of (38), as a funciotn of z with values in $B(H,H)$, is holomorphic on G_a, and its inverse belongs to $B(H,H)$ when $z = a$. According to Lemma 21, there is a neighborhood $G_{a\zeta} \subset G_a$ of a on which

$$z \longmapsto (I-(\zeta+\mu_a)R(-\mu_a, z))^{-1} \in B(H,H)$$

is holomorphic. Therefore, $\zeta \in \rho(T(z))$ for $z \in G_{a\zeta}$, and the holomorphy of

$$z \in G_{a\zeta} \longmapsto R(\zeta, z) = R(-\mu_a, z)(I-(\zeta+\mu_a)R(-\mu_a, z))^{-1} \in B(H,V)$$

follows.

<u>Theorem 24</u>. The set $\mathcal{G}$, defined by (37), is open in $\mathbb{C}^{n+1}$, and $(\zeta, z) \longmapsto R(\zeta, z) \in B(H,V)$ is holomorphic on $\mathcal{G}$.

<u>Proof</u>. Let $(\eta, a) \in \mathcal{G}$. This proof is essentially a careful repetition of the second part of Lemma 23, with η replacing $-\mu_a$. Lemma 23 implies that a neighborhood $G_{a\eta} \subset G$ of a can be found such that $\eta \in \rho(T(z))$ for $z \in G_{a\eta}$, and $z \longmapsto R(\eta, z) \in B(H,V)$ is holomorphic on $G_{a\eta}$. Analogous to (38), we have

$$(40) \qquad (T(z)-\zeta)R(\eta,z) = I - (\zeta-\eta)R(\eta,z) \quad \text{for} \quad z \in G_{a\eta}.$$

Now the right-hand side of (40), as a function of (ζ,z), with values in $B(H,H)$, is holomorphic on $\mathbb{C}\times G_{a\eta}$ and takes on the value I when $(\zeta,z) = (\eta,a)$. A consequence of Lemma 21 is that there is a neighborhood $\mathcal{G}_{\eta a} \subset \mathbb{C}\times G_{a\eta}$ of (h a) on which $(\zeta,z) \longmapsto (I-(\zeta-\eta)R(\eta,z))^{-1} \in B(H,H)$ is holomorphic. Hence, $\zeta \in \rho(T(z))$ for $(\zeta,z) \in \mathcal{G}_{\eta a}$, which implies that $\mathcal{G}$ is open. It follows that

$$(\zeta,z) \longmapsto R(\zeta,z) = R(\eta,z)(I-\zeta-\eta)R(\eta,z))^{-1} \in B(H,V)$$

is holomorphic on $\mathcal{G}_{\eta a}$. □

The enxt corollary states a condition on the family $\{\Phi(z) : z \in G\}$ which guarantees that the operator $T(z)$ is self-adjoint if $z \in G \cap \mathbb{R}^n$.

<u>Corollary 25</u>. Suppose that $\bar{z} = (\bar{z}_1,\ldots,\bar{z}_n) \in G$ whenever $z \in G$. If $\Phi(z)[v,\phi] = \overline{\Phi(\bar{z})[\phi,v]}$ for all ϕ and v in V and for $z \in G$, then $T(z)^* = T(z)$ for all $z \in G$, in which $T(z)^*$ is the adjoint of $T(z)$ as an operator on H.

Note that $R(\zeta,z)$ is compact as an operator on H, when $\zeta \in \rho(T(z))$, beacuse $R(\zeta,z) \in B(H,V)$ and V is compactly embedded in H by hypothesis. Consequently, the spectrum of $T(a)$ consists entirely of eigenvalues that have finite multiplicity and no finite accumulation point.

Recall that given $w \in H$, we want to determine the existence and the dependence upon z of the solution $\phi(z) \in V$ of

$$\Phi(z)[\phi(z),v] = (w,v)_H \quad \text{for all} \quad v \in V.$$

It follows from (36) that this is equivalent to solving $T(z)\phi(z) = w$. When 0 is not an eigenvalue $\phi(z) = R(0,z)w$. Lemma 23 yields

Theorem 26. $\phi : \{z \in G : 0 \in \rho(T(z))\} \longrightarrow V$ is holomorphic.

When 0 is a simple eigenvalue, we have the following result.

Theorem 26. Let $a \in G$ and suppose 0 is a simple eigenvalue of $T(a)$. Then there exists a neighborhood $G_a \subset G$ of a and two functions λ and $z \longmapsto P(z) \in B(H,V)$, which are holomorphic on G_a, such that $\lambda(a) = 0$, $\lambda(z)$ is a simple eigenvalue of $T(z)$, and $P(z)$ projects H onto the 1-dimensional eigenspace that corresponds to $\lambda(z)$. Furthermore, there exists another holomorphic function $z \in G_a \longmapsto R_2(0,z) \in B(H,V)$ such that

$$\phi(z) = \frac{1}{\lambda(z)}P(z)w + R_2(0,z)w \tag{41}$$

for all $z \in G_a$ for which $\lambda(z) \neq 0$.

The remainder of this section will be devoted to the proof of Theorem 27. The theory concerning the eigenvalue problem associated to $T(z)$ is well developed (cf. [5]). In fact, the form of $\phi(z)$ in (41) is a direct consequence of that theory. We repeat here many of these ideas in the process of proving the conclusions about analyticity.

Since the spectrum of $T(a)$ is a discrete set of eigenvalues having no finite accumulation point, a Jordan curve $\mathscr{C}$ can be drawn in $\rho(T(a))$ so as to enclose an open set in $\mathbb{C}$ containing 0 in its interior and the other eigenvalues in the exterior of its closure. Then $\mathscr{C}\times\{a\} \subset \mathscr{G}$, where $\mathscr{G}$ is the open set defined by (37). Hence, for each $\zeta \in \mathscr{C}$ there is a disc $D(\zeta,r(\zeta)) \subset \mathbb{C}$, $(r(\zeta) > 0$, is the radius) and a polydisc $\Delta(a,\rho(\zeta)) \subset \mathbb{C}^n$ such that

$$(\zeta,a) \in D(\zeta,r(\zeta))\times\Delta(a,\rho(\zeta)) \subset \mathscr{G}.$$

However, $\mathscr{C}$ is compact, so a finite set $\{\zeta_j \in \mathscr{C} : j = 1,\dots,k\}$ can be chosen such that $\{D(\zeta_j,r(\zeta_j)) : j = 1,\dots,k\}$ covers $\mathscr{C}$. Consequently, $\mathscr{C} \subset$

$\rho(T(z))$ for $z \in G'_a \equiv \bigcap_{j=1}^{k} \Delta(a, \rho(\zeta_j))$.

Next the operator $P(z)$ is defined for $z \in G'_a$ as a Riemann integral of $B(H,V)$-valued functions by

$$P(z) \equiv -\frac{1}{2\pi i} \int_{\mathscr{C}} R(\zeta, z) d\zeta. \tag{42}$$

It is shown in [5] (Theorems III-6.17 and VII-1.7) that $P(z)$ is a projection operator and that $P(a)$ maps H onto $M_1(a)$, the 1-dimensional eigenspace associated with the eigenvalue 0 of $T(a)$. Moreover, H can be decomposed as $H = M_1(z) + M_2(z)$ for $z \in G'_a$, in which

$$M_1(z) \equiv P(z)H \quad \text{and} \quad M_2(z) \equiv (I-P(z))H. \tag{43}$$

It is also true that $P : G'_a \longrightarrow B(H,V)$ is holomorphic because, by Theorem 24, the same is true of $R : \mathscr{C} \longrightarrow B(H,V)$. Since $\dim(M_1(a)) = 1$, and since $P : G'_a \longrightarrow B(H,H)$ is continuous (recall that V is continuously embedded in H), it follows from [5] (paragraphs I-§4.6 and IV-§3.4) that

$$\dim(M_1(z)) = 1 \quad \text{for} \quad z \in G'_a. \tag{44}$$

Now let $\psi_a \in M_1(a)$ and non-zero, and define G_a to be an open connected subset of G'_a such that $a \in G_a$ and $(P(z)\psi_a, \psi_a) \neq 0$ for $z \in G_a$. Next, define

$$\psi(z) \equiv P(z)\psi_a \quad \text{for} \quad z \in G_a, \tag{45}$$

and it follows that $\psi : G_a \longrightarrow V$ is holomorphic with $\psi(a) = \psi_a$.

It follows from paragraphs III-§5.6 and III-§6.1 of [5] that for $z \in G_a$,

$$P(z)v \in D(T(z)) \quad \text{for all} \quad v \in D(T(z)),$$

and

$$T(z)v \in M_k(z) \quad \text{for} \quad v \in M_k(z) \cap D(T(z)) \quad \text{and} \quad k = 1.$$

Thus $T_k(z) : M_k(z) \longrightarrow M_k(z)$ for $k = 1,2,$ and for $z \in G_a$ can be defined by $T_k(z) \equiv T(z)|_{M_k(z) \cap D(T(z))}$. The eigenvalue problem has now been decomposed into two eigenvalue problems, one for each $T_k(z)$ in $M_k(z)$. Of particular interest here is the eigenvalue problem for $T_1(z)$, because 0 and ψ_a form an eigenvalue-eigenfunction pair for $T_1(a)$.

According to (44), $T_1(z)$ is a 1-dimensional operator. Therefore, $\lambda(z) \equiv \text{trace}(T_1(z))$ is the eigenvalue of $T_1(z)$, i.e., for each $z \in G_a$,

(46) $$T(z)\psi_a = T_1(z)\psi_z = \lambda(z)\psi_z \quad \text{for all} \quad \psi_z \in M_1(z) \cap D(T(z)).$$

In (46), setting $\psi_z = P(z)\psi_a$ and taking inner products in H with ψ_a, yields

(47) $$\lambda(z) = \frac{(T(z)P(z)\psi_a, \psi_a)_H}{(P(z)\psi_a, \psi_a)_H} \quad \text{for} \quad z \in G_a.$$

Since $T(z)$ is a closed operator, it follows from (42) and the identity $T(z)R(\zeta,z) = I + \zeta R(\zeta,z)$ that $T(z)P(z) = -\frac{1}{2\pi i}\int_{\mathscr{C}} \zeta R(\zeta,z)d\zeta$. Consequently, $z \longmapsto T(z)P(z) \in B(H,V)$ is holomorphic on G_a $(\subset G_a')$. Therefore, the analyticity of λ on G_a follows from (47) because G_a was chosen so that $(P(z)\psi_a, \psi_a) \neq 0$ when $z \in G_a$. Since $\mathscr{C} \subset \rho(T(z))$ for all $z \in G_a$, $\lambda(z)$ lies in the interior of the open set enclosed by $\mathscr{C}$, whereas the remainder of the spectrum of $T(z)$ must lie in the open set that is exterior to $\mathscr{C}$.

Let $z \in G_a$ such that $\lambda(z) \neq 0$. Then $R(0,z)$ exists and commutes with $P(z)$. Consequently, we can define $R_k(0,z) \in B(H,V)$ for $k = 1,2,$ by

$$R_1(0,z) \equiv (R(0,z)P(z) = P(z)R(0,z),$$

and

$$R_2(0,z) \equiv R(0,z)(I-P(z)) = (I-P(z))R(0,z).$$

By passing $R(0,z)$ under the integral sign in (42), then using the resolvent equation to otbain $R(0,z) - R(\zeta,z) = -\zeta R(0,z)R(\zeta,z)$ when 0 and ζ are in $\rho(T(z))$, and noting that 0 lies inside the open set enclosed by $\mathscr{C}$,

$$R(0,z)P(z) = R(0,z) - \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{R(\zeta,z)}{\zeta}d\zeta \tag{48}$$

is obtained for each $z \in G_a$ such that $\lambda(z) \neq 0$. However, $\mathscr{C} \subset \rho(T(z))$ for all $z \in G_a$, so that the second term on the right-hand side of (48) is holomorphic as a function of z on G_a with values in $B(H,V)$. Consequently, $z \longmapsto R_2)9,z) \in B(H,V)$ can be continued analytically to all of G_a.

Finally, it is clear from (43) and the definition of $R_1(0,z)$ that $R_1(0,z)w \in M_1(z) \cap D(T(z))$ for all $w \in H$ and for each $z \in G$ such that $\lambda(z) \neq 0$. Then (46) implies

$$\lambda(z)R_1(0,z)w = T(z)R_1(0,z)w = T(z)R(0,z)P(z)w = P(z)w,$$

from which $R_1(0,z)w = \frac{1}{\lambda(z)}P(z)w$ is obtained from all $w \in H$ and for each $z \in G$ such that $\lambda(z) \neq 0$. This finishes the proof of Theorem 27 because $\phi(z) = R(0,z)w = R_1(0,z)w + R_2(0,z)w$ when $\lambda(z) \neq 0$.

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